

Schrödinger uncertainty relation and its minimization states *

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Abstract

An introductory survey on the Schrödinger uncertainty relation and its minimization states is presented with minimal number of formulas and some historical points. The case of the two canonical observables, position and momentum, is discussed in greater detail: basic properties of the two subsets of minimization states (canonical squeezed and coherent states) are reviewed and compared. The case of two non-canonical observables is briefly outlined. Stanford $SU(1,1)$ and $SU(2)$ group-related coherent states can be defined as states that minimize Schrödinger inequality for the three pairs of generators simultaneously. The symmetry of the Heisenberg and Schrödinger relations is also discussed, and two natural generalizations to the cases of several observables and several states are noted.

1 The Heisenberg uncertainty principle

The uncertainty (indeterminacy) principle is a basic feature of quantum physics. It reveals the fundamental difference between quantum and classical descriptions of Nature. The indeterminacy principle was introduced in 1927 by Heisenberg [1] who demonstrated the impossibility of simultaneous precise measurement of the canonical quantum observables \hat{x} and \hat{p}_x (the coordinate and the momentum) by positing an approximate relation $\delta p_x \delta x \sim \hbar$, where \hbar is the Plank constant, $\hbar = 1.05 \times 10^{-27}$ erg.sec. *"The more precisely is the position determined, the less precisely is the momentum known, and vice versa"* [1]. Heisenberg considered this inequality as the "direct descriptive interpretation" of the canonical commutation relation between the operators of the coordinate and momentum: $[\hat{x}, \hat{p}_x] = i\hbar$, $[\hat{x}, \hat{p}_x] \equiv \hat{x}\hat{p}_x - \hat{p}_x\hat{x}$. Qualitative statements about the repugnance of the precise determination of the coordinate with that of the momentum have been formulated in 1926 also by Dirac and Jordan (see refs. in M.Jammer, *The conceptual development of quantum mechanics*, Mc Graw-Hill, 1967). Let us recall that in quantum physics a physical quantity (observable) X is represented by a Hermitian operator \hat{X} in the Hilbert space of states. Soon after the Heisenberg paper [1] appeared Kennard and Weyl [2] proved the inequality

$$(\Delta p_x)^2 (\Delta x)^2 \geq \hbar^2/4, \quad (1)$$

where $(\Delta p_x)^2$ and $(\Delta x)^2$ are the variances (dispersions) of \hat{p}_x and \hat{x} , defined by Weyl for every quantum state $|\psi\rangle$ via the formula $(\Delta p_x)^2 := \langle \psi | (\hat{p}_x - \langle \psi | \hat{p}_x | \psi \rangle)^2 | \psi \rangle$, and similarly is $(\Delta \hat{x})^2$ defined. The matrix element $\langle \psi | \hat{X} | \psi \rangle \equiv \langle \hat{X} \rangle$ is the mean value of the observable \hat{X} in the state $|\psi\rangle$. The square-root $\Delta X = \sqrt{(\Delta \hat{X})^2}$ is called standard deviation.

In correspondence with the classical probability theory the standard deviation ΔX is considered as a measure for the indeterminacy (or for the fluctuations) of the observable \hat{X} in the corresponding state $|\psi\rangle$. In 1930 Dichburn [R.Dichburn, Proc. Royal Irish Acad. **39**, 73 (1930)] established the relation

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between Weyl' Δx and Heisenberg' δx [namely $\Delta x = \delta x/\sqrt{2}$], and proved that the equality $\delta p_x \delta x = h/2\pi$ can be achieved for Gauss probability distributions only. They are these distributions for which Heisenberg derived his relation, resorting to the properties of the Fourier transformation.

The Heisenberg-Weyl-Kennard inequality (1) became known as the *Heisenberg uncertainty relation*.

Generalization of (1) to the case of arbitrary two observables (Hermitian operators \hat{X} and \hat{Y}) was made by Robertson in 1929 [3],

$$(\Delta X)^2(\Delta Y)^2 \geq \frac{1}{4} \left| \langle [\hat{X}, \hat{Y}] \rangle \right|^2. \quad (2)$$

Robertson inequality (2) became known again as the Heisenberg uncertainty relation for two observables \hat{X} and \hat{Y} , and it is regarded as a mathematical formulation of the Heisenberg indeterminacy principle for two quantum observables. In view of this inertia and of the significant Robertson' contribution we shall refer to the relation (2) as Heisenberg–Robertson inequality or Heisenberg–Robertson uncertainty relation (while (1) is referred to as Heisenberg relation).

2 The Schrödinger inequality

The Heisenberg–Robertson uncertainty relation (2) and/or its particular case (1) became an irrevocable part of almost every textbook in quantum mechanics. However from the classical probability theory it is known that for two random quantities one can define three second moments: the variances of each observable and their *covariance*. In the relations (2) and (1) the two variances \hat{X} and \hat{Y} are involved only. This fact was first noted by Schrödinger in 1930 [4], who derived (using Schwartz inequality) the more general inequality

$$(\Delta X)^2(\Delta Y)^2 - (\Delta XY)^2 \geq \frac{1}{4} \left| \langle [\hat{X}, \hat{Y}] \rangle \right|^2, \quad (3)$$

where ΔXY denotes the covariance¹ of \hat{X} and \hat{Y} , $\Delta XY := (1/2)\langle \hat{X}\hat{Y} + \hat{Y}\hat{X} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle$. The ratio $r = \Delta XY / \Delta X \Delta Y$ is called correlation coefficient for two observables. In the classical probability theory the vanishing covariance is a necessary (but not sufficient) condition for the statistical independence of two random quantities. Nonvanishing covariance means stronger correlation between physical quantities.

In the case of coordinate and momentum observables relation (3) takes the shorter form of $(\Delta x)^2(\Delta p_x)^2 - (\Delta x p_x)^2 \geq \hbar^2/4$. For the sake of brevity henceforth we shall work with dimensionless observables $\hat{q} = \hat{x}\sqrt{m\omega/\hbar}$ and $\hat{p} = \hat{p}_x/\sqrt{m\omega\hbar}$ (instead of \hat{x} and \hat{p}_x), where m and ω are parameters with dimension of mass and frequency respectively. For \hat{q} and \hat{p} the Heisenberg and Schrödinger inequalities read simpler: $(\Delta q)^2(\Delta p)^2 \geq 1/4$,

$$(\Delta q)^2(\Delta p)^2 - (\Delta qp)^2 \geq \frac{1}{4}. \quad (4)$$

The Schrödinger inequality (3) is more general and more precise than that of Heisenberg–Robertson, eq. (2): the former is reduced to the latter in states with vanishing covariance of \hat{X} and \hat{Y} , and the equality in (2) means the equality in (3), the inverse being not true. Thus the Schrödinger inequality provides a more stringent limitation (from below) to the product of two variances. Despite of these advantages the relation (4) and/or (3) are lacking in almost all text books. The interest in Schrödinger relation has been renewed in the last two decades only (for the first time, to the best of my knowledge, in ref. [5] – 50 years after its discovery) in connection with the description and experimental realization of the *squeezed states* (SS) of the electromagnetic radiation [6, 7].

¹Other notations, used in the literature, for the covariance (variance), are $\text{Cov}(X, Y)$, σ_{XY} , $\Delta(X, Y)$ ($\text{Var}(X)$, σ_X^2 , ΔX^2 , $\Delta^2 X$). In his original paper [4] Schrödinger didn't introduce any symbol for the quantity $(1/2)\langle \hat{X}\hat{Y} + \hat{Y}\hat{X} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle$, while for the variance (mean-square deviation) he used $(\Delta X)^2$.

Another useful property of the Schrödinger uncertainty relation consists in its form-invariance against nondegenerate linear transformations of the two observables [9]. If one transforms \hat{X} , \hat{Y} to \hat{X}' , \hat{Y}' ,

$$\hat{X}' = \lambda_{11}\hat{X} + \lambda_{12}\hat{Y}, \quad \hat{Y}' = \lambda_{21}\hat{X} + \lambda_{22}\hat{Y},$$

one would obtain that the left and the right hand sides of (3) are transformed in the same manner (*covariantly*) – by multiplication by the factor $(\det \Lambda)^2$, where Λ is 2×2 matrix of the transformation coefficients λ_{ij} . It then follows that if a given state saturates (3) for \hat{X} and \hat{Y} , it saturates (3) for the transformed observables \hat{X}' , \hat{Y}' as well. The covariance of the Heisenberg–Robertson inequality, eq. (2), is much more restricted: both its sides are transformed in the same manner under the linear scale transformations of the observables $\hat{X}' = a\hat{X}$, $\hat{Y}' = b\hat{Y}$ *only*. In the case of canonical observables this restricted covariance means that the equality in (1) is not invariant under linear canonical transformations, in particular it is not invariant under rotations in the phase plane.

The scale transformations and the rotations form two distinct subgroups of the group of linear canonical transformations. It is quite natural then to look for another uncertainty relation for q and p which is covariant (the equality being invariant) under the rotations in the phase space. It turned out that such covariant inequality does exist: it has the simple form of

$$(\Delta q)^2 + (\Delta p)^2 \geq 1. \quad (5)$$

This inequality is less precise than (4) and (1) in the sense that the equality in it entails the equality in both (4) and (1). The equality in (5) is invariant under rotations in the phase plane.

The most precise inequality, the Schrödinger one, eq. (4), is most symmetric – the equality in (4) is invariant under both subgroups (rotations and scale transformations) of the group of linear canonical transformations of \hat{q} and \hat{p} . For two arbitrary observables (Hermitian operators) the inequality (5) takes the form $(\Delta X)^2 + (\Delta Y)^2 \geq |\langle [\hat{X}, \hat{Y}] \rangle|$.

3 Minimization of the uncertainty relations

The interest in minimization of uncertainty relations has increased after the discovery of *coherent states* (CS) of the electromagnetic field in 1963 (independently by the american physicists Klauder, Glauber and Sudarshan – see refs. in [11]), and especially after the discovery of *squeezed states* (SS) [6, 7]. Next we consider the basic properties of states, that minimize Heisenberg and Schrödinger inequalities for the canonical pair \hat{p} , \hat{q} (subsections **A** and **B**) and for some non-canonical observables: the spin and quasi-spin components (subsection **C**).

States which minimize the Heisenberg–Robertson inequality (2) have been called *intelligent* (C.Aragon et al, J.Math.Phys. **17** (1976) 1963), while those which minimize the more general Schrödinger inequality (3) were named *correlated* [5] and *generalized intelligent* states [9]. The names Heisenberg (Schrödinger) intelligent states, Heisenberg (Schrödinger) minimum uncertainty states and Heisenberg (Schrödinger) optimal uncertainty states are also used (see the review papers D.A.Trifonov, JOSA A **17**, 2486 (2000) (e-print quant-ph/0012072); e-print quant/9912084 and refs. therein).

A. Minimization of the Heisenberg inequality

As it could be seen from (3) and (2) the problem of minimization of the Heisenberg–Robertson relation is a particular case of the minimization of the Schrödinger inequality, corresponding to vanishing correlation coefficient of the two observables. In fact it was Heisenberg [1] who has first minimized the inequality (1), showing that for Gaussian distribution of x of the form $\exp[-(x - x')^2/(\delta x)^2]$ the equality $\delta x \delta p_x = h/2$ holds. The problem of minimization of inequalities (2), (4) or (3) *was not considered* in Robertson and Schrödinger papers [3, 4].

Glauber CS. Most widely known minimizing states are the CS of the electromagnetic field [11], which are considered as states of the *ideal monochromatic laser radiation*. These states (called Glauber-Klauder-Sudarshan CS, Glauber CS, or canonical CS) are defined as normalized eigenstates of the non-Hermitian photon annihilation operator \hat{a} , $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where the complex number α is the eigenvalue of the operator \hat{a} . The CS $|\alpha\rangle$ possess several remarkable physical

and mathematical properties, which led to their intensive applications in many fields of modern theoretical and mathematical physics.

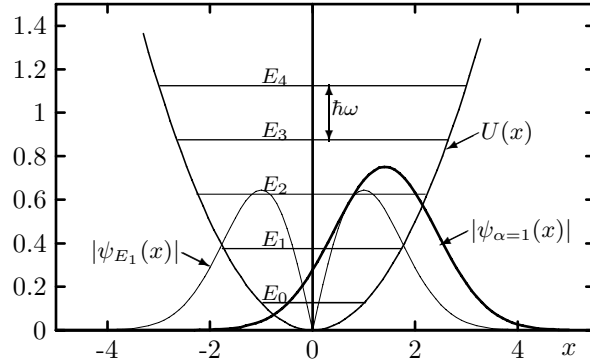
Before describing the properties of the Glauber CS it would be useful to recall the *stationary states* $|n\rangle$ (with corresponding wave function $\psi_n(x)$) of the harmonic oscillator. Stationary states are defined for every quantum system as states with definite energy, i.e. as eigenstates of the energy operator (the Hamiltonian) \hat{H} . For the mass oscillator (a particle with mass m in the parabolic potential well $U(x) = m\omega^2 x^2/2$) the Hamiltonian is $\hat{H} = \hat{p}^2/2m + m\omega^2 x^2/2$. Its eigenvalues (the energy levels) are discrete and equidistant, $E_n = \hbar\omega(n + 1/2)$, $n = 0, 1, \dots$. The distance between neighbor levels is equal to $\hbar\omega$.

The transition from $|n\rangle$ to $|n-1\rangle$ is performed by the action of the operator $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$ on $|n\rangle$, and the transition to $|n+1\rangle$ – by the action of the conjugated operator $\hat{a}^\dagger = (\hat{q} - i\hat{p})/\sqrt{2}$. This shows that \hat{a} and \hat{a}^\dagger can be regarded as operators, that annihilate and create photon with Planck energy $\hbar\omega$, $\hat{a}^\dagger \hat{a}$ – as operator of the number of photons, and $|n\rangle$ – as a state with n photons. In the context of the electromagnetic field however \hat{q} and \hat{p} do not have the meaning of coordinate and moment. For the field in the one-dimensional cavity \hat{q} is proportional to the electric intensity, and \hat{p} – to the magnetic intensity (Loudon and Knight [7]).

The time evolution $|n; t\rangle$ of an initial state $|n\rangle$ is $|n; t\rangle = \exp(-iE_n t/\hbar)|n\rangle$. This form shows that the probability distribution density of the coordinate $|\psi_n(x, t)|^2$ is static: $|\psi_n(x, t)| = |\psi_n(x)|$. The energy levels E_n and graphics of $U(x)$ and $|\psi_1(x)|$ for an oscillator with frequency $\omega = 1/4$ and mass $m = 4$ are shown on figure 1. In every state $|n\rangle$ the mean coordinate and the mean moment are equal to zero. The states $|n\rangle$ are orthonormalized and form a basis in the Hilbert space of states, which means that any other state is a superposition of stationary states.

In Glauber CS $|\alpha\rangle$ the covariance of the canonical pair of observables \hat{p} , \hat{q} vanishes, and the variances of \hat{q} and \hat{p} are equal: $(\Delta q)^2 = 1/2$, $(\Delta p)^2 = 1/2$. These two moments minimize the Heisenberg inequality. Due to this inequality the value of $1/2$ is the minimal possible one that two dispersions $(\Delta p)^2$ and $(\Delta p)^2$ can take simultaneously. The CS $|\alpha\rangle$ are the only states with this property: In any other state at least one of the two dispersions is greater than $1/2$. This fact means that in CS $|\alpha\rangle$ the trajectory of the mass oscillator in the phase space is determined with the highest possible accuracy. Correspondingly, in the context of electromagnetic field we have that in CS $|\alpha\rangle$ the fluctuations of the electric and magnetic intensities are minimal.

Figure 1. Energy levels E_n , potential energy $U(x)$ and absolute values of wave functions of stationary state $|\psi_n(x)|$ with $n = 1$ (two maximums) and of CS $|\psi_\alpha(x)|$ with $\alpha = 1$ (one Gauss maximum) for the harmonic oscillator. The graphics of $|\psi_n(x)|$ is static, while the maximum of $|\psi_\alpha(x)|$ is oscillating harmonically exactly as the classical particle oscillates.



In CS the mean value of the coordinate coincides with the most probable one and (for the stationary oscillator) depends on t harmonically, exactly as the coordinate of classical particle depends on t . In this sense the quantum states $|\alpha\rangle$ are "*most classical*". CS $|\alpha\rangle$ have the form of an infinite superposition of $|n\rangle$, the mean energy being equal to $\hbar\omega(|\alpha|^2 + 1/2)$. Graphics of the absolute value of the wave function $|\psi_\alpha(x, t)|$ (the square root of the probability density) with $\alpha = 1$ and $t = 2k\pi/\omega$, $k = 0, 1, \dots$ is shown on figure 1.

CS $|\alpha\rangle$ possess other "classical properties" as well: minimal energy of quantum fluctuations, Poisson photon distribution, $[P_n(\alpha) = |\alpha|^{2n} \exp(-|\alpha|^2)/n!]$, and positive Wigner and Glauber-Sudarshan quasi-distributions. The last property enables one to represent correctly the quantum-mechanical mean values as classical mean values of the corresponding classical quantities. However, one can show that in CS $|\alpha\rangle$ all observables fluctuate, i.e. these states are not eigenstates of any Hermitian operator.

It is worth noting the physical meaning of the eigenvalue property of CS $|\alpha\rangle$ (eigenstates of the ladder operator \hat{a}): the annihilation of one photon in CS doesn't change, up to a normalization

constant, the state of the field. In particular the mean energy (in $|\alpha\rangle$ and in the normalized $\hat{a}|\alpha\rangle$) remains the same. Destruction of n photons also doesn't change the state, since $|\alpha\rangle$ is an eigenstate of any power of \hat{a} . This remarkable property is typical for *infinite* superpositions of $|n\rangle$ *only*, as the CS $|\alpha\rangle$ are. It is the "infinity" that compensates the annihilation of the n photons. Unlike \hat{a} , the creation operator \hat{a}^\dagger has no eigenstate at all, and the photon added states $\hat{a}^\dagger|\alpha\rangle$ are no more CS. All ladder operators in *finite* dimensional Hilbert space have no eigenstates (except for the state, that is annihilated by the ladder operator).

From the remarkable mathematical properties of the canonical CS we will note here their "over-completeness" and "orbitality". The first property means that the family of CS $|\alpha\rangle$ is overcomplete in the Hilbert space, i.e. any other state $|\psi\rangle$ can be represented as a continuous superposition of CS: $|\psi\rangle = (1/\pi) \int |\alpha\rangle \langle\alpha|\psi\rangle d^2\alpha$. This overcompleteness enables one to represent states as analytic functions of α (or as functions in the phase space respectively), and abstract operators – as differential operators. For example, $\hat{a} = d/d\alpha$ and $\hat{a}^\dagger = \alpha$. This *CS representation* provides the possibility to use powerful analytic method in treating various problems of quantum physics. It is very convenient in elucidating the relationship between quantum and classical description of physical systems.

The orbitality property consists in the fact, that the family of CS $|\alpha\rangle$ is an orbit of the unitary Weyl displacement operators $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ through the ground state $|0\rangle$, i.e. $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$. One says that Glauber CS are generated from the vacuum $|0\rangle$ by the action of Weyl operators. The set of Weyl operators form an unitary representation of the group of Heisenberg-Weyl. As early as in 1963 Klauder suggested that overcomplete families of states could be constructed using unitary representations of other Lie groups. For the group of rotations $SO(3)$ such overcomplete family was constructed in 1971 by Radcliffe (spin CS), and for the group of pseudo-rotations $SO(1,2)$ – by Solomon in 1971 (A. Solomon, J.Math.Phys. **12**, 390 (1971)) and Perelomov in 1972 (quasi-spin CS) (see refs. in [11]). Perelomov proved that orbits of operators of irreducible unitary representations of any Lie group do form overcomplete sets of states. The quantum evolution operators of system with Lie group symmetry are operators of the corresponding unitary representations of the group. This gives the idea how to generate physically new families of states starting from known initial ones.

General form of states, minimizing Heisenberg inequality. Glauber CS are not the most general ones that minimize the Heisenberg relation (1). Evidently, if in (1) one variance increases κ -times, and the other decreases κ -times, the equality in (1) will be preserved. Such change in the variances can be achieved acting on CS $|\alpha\rangle$ by the unitary operator $\hat{S}(s) = \exp[s(\hat{a}^{\dagger 2} - \hat{a}^2)/2]$, where s is real parameter [8]. In states $|\alpha, s\rangle = \hat{S}(s)|\alpha\rangle$ we have $\Delta p = e^{-s} \frac{1}{\sqrt{2}}$ and $\Delta q = e^s \frac{1}{\sqrt{2}}$. One sees that for $s \neq 0$ one of the variances is decreased below the value of $1/2$. States in which Δq or Δp is below $1/\sqrt{2}$ (the value of Δq and Δp in the CS), have been called *squeezed states*.

States $|\alpha, s\rangle$ are the most general ones that minimize Heisenberg inequality. However these states are *extremely unstable* in time (D.Stoler, Phys.Rev. **D11**, 3033 (1975); D.Trifonov, Phys.Lett. **A48** (1974) 165). If the oscillator is prepared at $t = 0$ in a state $|\alpha, s\rangle$ with $s \neq 0$, then at $t > 0$ it goes out of the family $\{|\alpha, s\rangle\}$ and the equality in (1) is violated. Moreover, in the evolved oscillator state $|\alpha, s; t\rangle$ inevitably the covariance of \hat{q} and \hat{p} is generated, which is not taken into account in the Heisenberg relation. In the Heisenberg picture the free oscillator evolution operator acts as rotation on angle $-\omega t$ in the phase space. These rotations do not preserve the initial form of the relation (1).

Unlike the Heisenberg relation (1), the inequality (5) is invariant under rotations in phase plane, and this is another explanation of the temporal stability of $|\alpha\rangle$ under free oscillator evolution. It is the uncertainty relation (5) that is minimized in CS $|\alpha\rangle$ only: in any other state $(\Delta q)^2 + (\Delta p)^2 > 1$.

B. Minimization of the Schrödinger inequality. Squeezed states

The problem of minimization of the Schrödinger inequality (4) was first considered (as far as I know) in 1980 in [5], where it was shown that eigenstates $|\beta, \lambda\rangle$ of the operator $\lambda\hat{q} + i\hat{p}$, minimize (4) for every complex λ and β . At $\text{Im}\lambda = 0$ one has $\Delta qp = 0$ and the solutions $|\beta, \lambda\rangle$ coincide with the

above $|\alpha, s\rangle$. For complex λ the states $|\beta, \lambda\rangle$ turned out to coincide [9] with the Stoler states $|\alpha, \zeta\rangle$ introduced providently in his 1970 paper [8]: $|\alpha, \zeta\rangle = S(\zeta)|\alpha\rangle$, where $S(\zeta) = \exp[(\zeta\hat{a}^{\dagger 2} - \zeta^*\hat{a}^2)/2]$, $\zeta \in \mathbf{C}$. The family $\{|\alpha, \zeta\rangle\}$ received broad popularity as the family of *squeezed states* (called also canonical squeezed states), and the unitary operator $S(\zeta)$ became known as *squeeze operator* [7].

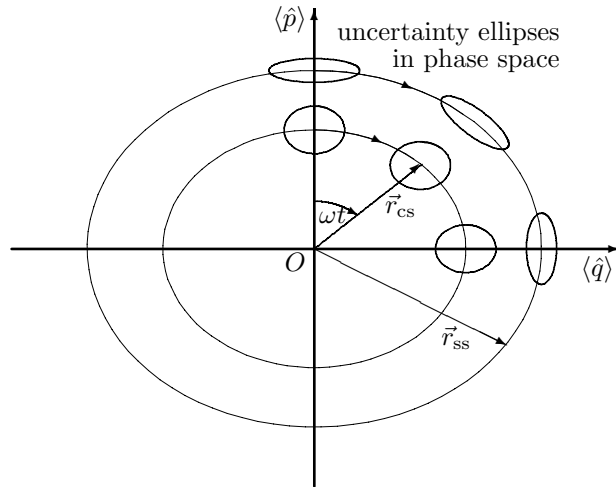
The canonical SS can be defined [5, 9] as states that minimize Schrödinger uncertainty relation (4), i.e. as solutions to the equation

$$(\mu\hat{a} + \nu\hat{a}^{\dagger})|\alpha, \mu, \nu\rangle = \alpha|\alpha, \mu, \nu\rangle. \quad (6)$$

where μ and ν are *complex* parameters, and $|\mu|^2 - |\nu|^2 = 1$. It is the equation (6) where the alternative notation (the Yuen notation – see ref. in [7]) $|\alpha, \mu, \nu\rangle$ for the squeezed states stems from. For Stoler states $|\alpha, \zeta\rangle$ one has $\mu = \text{ch}|\zeta|$, $\nu = -\text{sh}|\zeta|\exp(i \arg \zeta)$. At $\nu = 0$ the canonical CS $|\alpha\rangle$ are reproduced. The free oscillator evolution of SS $|\alpha, \mu, \nu\rangle$ is stable (the equality in (4) is preserved by any Hamiltonian at most quadratic in \hat{p} , \hat{q} [9]). For the electromagnetic field these states are experimentally realized, the corresponding light being called squeezed [7]. The SS $|\alpha, \mu, \nu\rangle$ can be generated by letting the laser light (which is supposed to be in a Glauber CS $|\alpha\rangle$) pass through nonlinear optical media. The simplest optical nonlinear interaction is described by a Hamiltonian which is a linear combination of squared \hat{a} and \hat{a}^{\dagger} . The quantum evolution operator, corresponding to such quadratic interaction takes exactly the form of the squeeze operator $S(\zeta(t))$, i.e. the evolved CS is of the form of SS $|\alpha, \zeta\rangle$.

The wave function $\psi_{\alpha}(x; \mu, \nu)$ of SS $|\alpha, \mu, \nu\rangle$ is a quadratic in terms of x exponent, a particular case of which is the wave function of CS $|\alpha\rangle$. This implies that some of the properties of the two kinds of states should be similar. An examples of such similar properties are the circular form of the oscillator phase space trajectories (see Fig. 2), and the coincidence of the mean coordinate and moment with the most probable ones. Let us recall that in stationary states the trajectory is degenerated into a point and the mean coordinate and moment deviate from the most probable ones.

Figure 2. Trajectories of the mean values of \hat{p} and \hat{q} and uncertainty ellipses in CS $|\alpha; t\rangle$ (a circle with radius r_{cs}) and in SS $|\alpha, \mu, \nu; t\rangle$ (a circle with radius r_{ss}) of the free oscillator. The means are oscillating with frequency ω , and the variances – with 2ω . At $\alpha = 0$ (vacuum and squeezed vacuum) the trajectories are degenerated into a point ($r_{cs}=0=r_{ss}$).



The free oscillator time-evolution preserves Schrödinger intelligent states stable : $|\alpha, \mu, \nu; t\rangle = |\alpha(t), \mu(t), \nu(t)\rangle$, $\mu(t) = \mu \exp(i\omega t)$, $\nu(t) = \nu \exp(-i\omega t)$. The centers of the wave packets of the time-evolved states $|\alpha(t), \mu(t), \nu(t)\rangle$ and CS $|\alpha(t)\rangle$ oscillates with the same period $2\pi/\omega$ (while in $|n; t\rangle$ the picture is static). Fluctuations in \hat{q} and \hat{p} in $|\alpha(t), \mu(t), \nu(t)\rangle$ are oscillating in time with frequency 2ω , their sum remaining constant: $(\Delta q)^2(t) = |\mu(t) - \nu(t)|^2/2$, $(\Delta p)^2(t) = |\mu(t) + \nu(t)|^2/2$. These two variances, the corresponding covariance $\Delta qp(t) = \text{Im}[\mu^*(t)\nu(t)]$ and the mean values $\langle \cdot \rangle_t$ of q and p in the evolved state determine an ellipse in the phase space,

$$(\Delta p)^2(t) (q - \langle q \rangle_t)^2 + (\Delta q)^2(t) (p - \langle p \rangle_t)^2 - 2\Delta qp(t) (q - \langle q \rangle_t)(p - \langle p \rangle_t) = (\Delta p)^2(t_0)(\Delta q)^2(t_0),$$

where the initial moment t_0 is chosen such that $\Delta qp(t_0) = 0$. This ellipse is known as the *uncertainty ellipse*. It is also called ellipse of equal probabilities or Wigner ellipse, since Wigner quasi-distribution of $|\alpha, \mu, \nu\rangle$ is constant on it. The semiaxes of the Wigner ellipse are just the initial standard deviations $\Delta q(t_0)$ and $\Delta p(t_0)$. Note that the current variances are $(\Delta q)^2(t) = (\Delta q)^2(t_0) \cos^2(\omega t) + (\Delta p)^2(t_0) \sin^2(\omega t)$, $(\Delta p)^2(t) = (\Delta q)^2(t_0) \sin^2(\omega t) + (\Delta p)^2(t_0) \cos^2(\omega t)$, and these are not equal to projections of the uncertainty ellipse on to the coordinate axes.

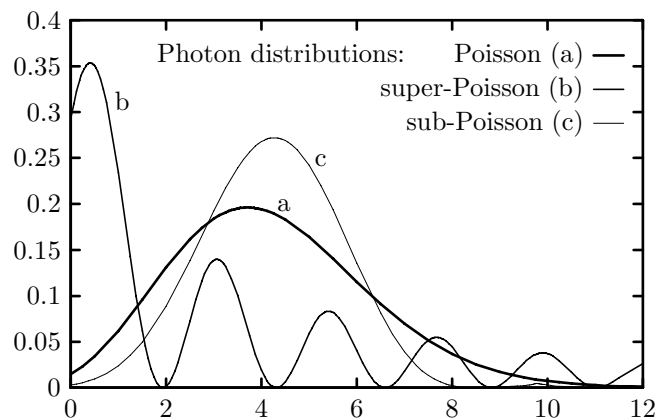
At $t = t_0$ the semiaxes are parallel to the coordinate axes. At $t > t_0$ they are rotated to an angle of $\varphi = -\omega t$. When $\varphi = 0, \pi, \dots$ the covariance $\Delta qp(t) = 0$, that is the covariance $\Delta qp(t)$ determines the inclination of the ellipse axes to the coordinate axes. Thus the free field time-evolution rotates the Wigner ellipse, preserving the length of its semiaxes (therefore its area is also constant) (see Fig. 3). The (stationary and nonstationary) oscillator time-evolution of the variances of \hat{q} and \hat{p} in Gaussian wave packets has been studied as early as in 1974 (M.Sargent, M.Scully and W.Lamb, *Laser physics*, Addison-Wesley, 1974; D.Trifonov, Phys.Lett. A48 (1974) 165).

Despite of the functional closeness of the wave functions of SS $|\alpha, \mu, \nu\rangle$ and CS $|\alpha\rangle$ some of their physical properties are significantly different. The main difference consists in the squeeze-effect, where the name SS for $|\alpha, \mu, \nu\rangle$ originates from: at $\nu \rightarrow \mu$ ($\nu \rightarrow -\mu$) the fluctuations of \hat{q} (of \hat{p}) in $|\alpha, \mu, \nu\rangle$ decrease below their value in CS and tend to zero (ideal squeezing – ideal SS). Note however that when $\text{Re}(uv^*) = 0$ there is no squeezing – both variances are greater than $1/2$.

Another difference, that was intensively discussed in the literature is the non-positivity of the Glauber-Sudarshan quasi-probability distribution $P(\alpha')$ (defined by means of $\hat{\rho} = \int P(\alpha') |\alpha'\rangle \langle \alpha'| d^2\alpha'$, where $\hat{\rho}$ is pure or mixed state). For CS $|\alpha\rangle$ one has $P(\alpha') > 0$, while for $|\alpha, \mu, \nu\rangle$ the function $P(\alpha')$ may be negative over some range of α' (see [7] and refs. therein). Due to violation of the positivity of this quasi-probability $|\alpha, \mu, \nu\rangle$ became known as *non-classical states*. This non-classicality is closely related to the squeeze-effect: reduction of the fluctuations of \hat{q} or \hat{p} in a given quantum state is a sufficient condition for the non-positivity of $P(\alpha')$. The third difference between CS and SS is related to their photon distributions. The Poisson photon distribution in the laser radiation (described by the CS $|\alpha\rangle$) is considered as a classical one. Its main feature is the equality of photon number variance with the mean number of photons. This equality is violated in states $|\alpha, \mu, \nu\rangle$ with $\nu \neq 0$. Number distribution with $(\Delta n)^2 > \langle \hat{n} \rangle$ is called super-Poissonian, and with $(\Delta n)^2 < \langle \hat{n} \rangle$ – sub-Poissonian [7]. Examples of sub- and super-Poissonian distributions in states $|\alpha, \mu, \nu\rangle$ are shown on figure 3 (graphics b and c).

Again there is a relation to the non-positivity of the Glauber-Sudarshan quasi-distribution: the latter is non-positive definite, if $(\Delta n)^2 < \langle \hat{n} \rangle$. Due to this property states with sub-Poissonian statistics are considered as non-classical. Experimentally the sub-Poissonian statistics is revealed as *photon antibunching* (impossibility to detect photons in arbitrary closed moments of time). Ideal photon antibunching (or maximal non-classicality) is exhibit in the states $|n\rangle$ with definite number of photons, for which $\Delta n = 0$, $\langle \hat{n} \rangle = n$. In contradistinction to the Poisson case the sub- and super-Poisson distributions may strongly oscillate.

Figure 3. Photon distributions in Schrödinger intelligent states $|\alpha, \mu, \nu\rangle$ with one and the same mean number of photons $\langle \hat{a}^\dagger \hat{a} \rangle \simeq 4.22$. Oscillations are typical to states with $\nu \neq 0$. In the cases (b) and (c) oscillations occur for large n , $n \rightarrow \infty$, as well, but in (c) they are invisible in the scale used.



Examples of oscillating sub- and super-Poisson photon distributions in Schrödinger intelligent states $|\alpha, \mu, \nu\rangle$ are shown in Fig. 3.

C. Minimization states for noncanonical observables

An *important application* of the states $|\alpha, \mu, \nu\rangle$, which minimize the inequality (4) for canonical pair $\hat{p}-\hat{q}$, was pointed out by Caves in 1981 [6], who analyzed the accuracy of the interferometric measurements of weak signals, such as the detection of the gravitational waves. He found, that the measurement accuracy can be significantly increased if laser light used in the interferometer is replaced with squeezed light, described by $|\alpha, \mu, \nu\rangle$. This fact motivated the search of squeezed states for other pairs of (noncanonical) observables \hat{X}, \hat{Y} , i.e. of field states with reduced quantum fluctuations of \hat{X} or \hat{Y} . In paper [9] it was proposed to construct SS for two general observables as states, that minimize Schrödinger inequality (3). Such minimizing states were called [9] \hat{X} - \hat{Y} generalized intelligent states, or \hat{X} - \hat{Y} Schrödinger intelligent states. Following [5] they could be called \hat{X} - \hat{Y} correlated states. The inequality (3) is minimized in a state $|\psi\rangle$ if and only if $|\psi\rangle$ is an eigenstate of (generally *complex*) combination of \hat{X} and \hat{Y} , i.e. if $|\psi\rangle$ satisfies the equation [9]

$$(u\hat{A} + v\hat{A}^\dagger)|\psi\rangle = z|\psi\rangle, \quad (7)$$

where $z, u, v \in \mathbf{C}$, $\hat{A} = \hat{X} + i\hat{Y}$. This equation shows that at $v \rightarrow u$ ($v \rightarrow -u$) the solution $|\psi\rangle \equiv |z, u, v\rangle$ must tend to the eigenstate of \hat{X} (of \hat{Y}), i.e. at $v \rightarrow u$ ($v \rightarrow -u$) quantum fluctuations of \hat{X} (\hat{Y}) must tend to zero. Therefore the solutions $|z, u, v\rangle$ to (7), when exist, are ideal squeezed states for the corresponding two observables (shortly \hat{X} - \hat{Y} SS).

The equation (7) was solved [9] for the pairs of Hermitian generators (observables) K_1, K_2 and J_1, J_2 of the groups $SU(1,1)$ and $SU(2)$. It turned out that in these cases the family of solutions $|z, u, v; k\rangle$ ($|z, u, v; j\rangle$) contains the standard $SU(1,1)$ and $SU(2)$ CS [11], i.e. all these group-related CS minimize the Schrödinger uncertainty relation for the first pair of the group generators. Generators of $SU(1,1)$ and $SU(2)$ have important boson/photon realizations (see, e.g. quant-ph/9912084 and refs. therein). For example, the $SU(1,1)$ generators can be realized by means of one pair of boson annihilation and creation operators \hat{a}, \hat{a}^\dagger as

$$\hat{K}_1 = \frac{1}{4}(\hat{a}^2 + \hat{a}^{\dagger 2}), \quad \hat{K}_2 = \frac{i}{4}(\hat{a}^2 - \hat{a}^{\dagger 2}), \quad \hat{K}_3 = \frac{1}{4}(2\hat{a}^\dagger\hat{a} + 1).$$

States $|z, u, v; k\rangle$ for this realization (here $k = 1/4, 3/4$) can exhibit squeezing in fluctuations not only of \hat{K}_1 and \hat{K}_2 , but also of \hat{q} and \hat{p} . In a certain range of parameters u and v squeezing may occur for \hat{K}_1 and \hat{p} (or for \hat{K}_2 and \hat{q} simultaneously (joint squeezing for two noncommuting observables). Under free field evolution the states $|z, u, v; k\rangle$ and $|z, u, v; j\rangle$ remain stable (as the canonical SS $|\alpha, \mu, \nu\rangle$ do). This means that if the electromagnetic radiation is prepared in such states, it would propagate stable in vacuum, and in a linear and homogeneous media as well. *Schemes for generation* of one-mode and two-mode light in states of the families $\{|z, u, v; k\rangle\}$ and $\{|z, u, v; j\rangle\}$ are proposed in several papers. Brif and Mann [C.Brif and A.Mann, Phys. Rev. **A54**, 4505 (1996)] showed that, light in states $|z, u, v; k\rangle$ (for \hat{K}_2 - \hat{K}_3) and in $|z, u, v; j\rangle$ (for \hat{J}_2 - \hat{J}_3) could be used for further significant increase of the accuracy of the interferometric measurements.

Schrödinger intelligent states are constructed analytically for every pair of the quasi-spin (K_i - K_j) and spin (J_i - J_j) components (see e.g. quant-ph/9912084; JOSA A **17** (2000) 2486, and refs. therein). Are there states of systems with $SU(1,1)$ ($SU(2)$) symmetry that minimize Schrödinger uncertainty relation for all the three pairs K_1 - K_2 , K_2 - K_3 and K_3 - K_1 (J_1 - J_2 , J_2 - J_3 and J_3 - J_1) *simultaneously*? The answer to this question is positive: such states with optimally balanced fluctuations of the three observables K_1, K_2, K_3 (J_1, J_2, J_3) (states with maximal $su(1,1)$ or $su(2)$ intelligency) are the known Klauder-Perelomov $SU(1,1)$ CS (Radcliffe-Gilmore $SU(2)$ CS) *only* (proof in quant-ph/9912084 and in J. Phys. A **31** (1998) 8041). For the above noted one-mode realization of the $su(1,1)$ these group related CS coincide with the known squeezed vacuum states and in our notations here they are $|\alpha = 0, \mu, \nu\rangle$. In the previous subsection we have seen that $|\alpha = 0, \mu, \nu\rangle$ minimize the Schrödinger relation for \hat{q} and \hat{p} as well. Hence the squeezed vacuum states are the unique states that minimize Schrödinger inequalities for four pairs of observables K_i - K_j and q - p

simultaneously. They are these states that was used by Caves [6] to increase the accuracy of the interferometric measurements.

4 Generalizations of the Schrödinger uncertainty relation

The Heisenberg and Schrödinger uncertainty relations reveal quantitatively the statistical correlations between two observables in one and the same state. Two natural questions can be immediately formulated: are there statistical correlations a) between several observables in one state? b) between observables in two and more states?

The positive answer to the first question was given by Robertson in 1934 [Phys. Rev. **46** 794 (1934)] by proving of the inequality

$$\det \sigma \geq \det C, \quad (8)$$

where σ is the matrix of all second moments (the uncertainty, the covariance or dispersion matrix) of n observables $\hat{X}_1, \dots, \hat{X}_n$, and C is the matrix of mean values of their commutators, $\sigma_{jk} = \langle X_j X_k + X_k X_j \rangle / 2 + \langle X_j \rangle \langle X_k \rangle \equiv \Delta X_j X_k$, $C_{jk} = -(i/2) \langle [X_j, X_k] \rangle$. At $n = 2$ Robertson uncertainty relation (8) coincides with the Schrödinger one, eq. (3). The minimization of Robertson inequality (8) is considered in [10].

The second question also has a positive answer [D.A.Trifonov, J. Phys. **A33** (2000) L296]. Here is the invariant generalization of the Schrödinger relation (4) for \hat{p} and \hat{q} to the case of two states $|\psi\rangle$ and $|\phi\rangle$,

$$\frac{1}{2} [(\Delta_\psi q)^2 (\Delta_\phi p)^2 + (\Delta_\phi q)^2 (\Delta_\psi p)^2] - |\Delta_\psi q p \Delta_\phi q p| \geq \frac{1}{4}, \quad (9)$$

where $(\Delta_\psi q p)^2$ is the covariance of \hat{q} and \hat{p} in the state $|\psi\rangle$. At $|\psi\rangle = |\phi\rangle$ this inequality reproduces that of Schrödinger, eq. (4). The relation (9) is neither a sum nor a product of the two Schrödinger relations for $|\psi\rangle$ and $|\phi\rangle$ correspondingly. It can not be represented as a sum or as a product of two quantities, each one depending on one of the two states only. Such unfactorizable uncertainty relations are called *state entangled*. The inequality (9), and (4) as well, contains the second statistical moments of \hat{q} and \hat{p} , which are measurable quantities. The experimental verification of the relation (9) would be, we hope, a new confirmation of the Hilbert space model of quantum physics.

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